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BALANCING MATRICES WITH LINE SHIFTS

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Dedicated to Paul Erdős on his seventieth birthday

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We give a purely deterministic proof of the following theorem of J. Komlós and M. Sulyok. Let $A=(a_{ij})$, $a_{ij}=\pm 1$ be an $n\times n$ matrix. One can multiply some rows and columns by -1 such that the absolute value of the sum of the elements of the matrix is $\leqq 2$ if n is even and 1 if n is odd. Note that Komlós and Sulyok applied probabilistic ideas and so their method worked only for $n>n_0$.

Consider a square $n\times n$ array of lights. Each light has two possible states, on and off. To each row and to each column of the array there is a switch. Turning this switch changes the state of each light in that row and column.

This concept is attributed to David Gale. In the late 1960's Elwyn Berlekamp built such a machine (of size 8×8) and it was a fixture in the tea room of the Mathematics Department at Bell Telephone Laboratories (Murray Hill) for many years.

By a configuration we shall mean the state of a given machine, i.e. precisely which lights are on and off. Define the signed discrepancy of a configuration to be the number of lights on minus the number of lights off and define the discrepancy to be the absolute value of the signed discrepancy. We shall consider the following solitaire game: Given an initial configuration the Player switches rows and columns in order to minimize the discrepancy. (Other games, such as attempting to maximize the discrepancy, have also been examined [1] in detail.)

When n is odd the discrepancy is always at least one. When n is even the signed discrepancy remains in the same residue class modulo 4 under line shifts. If in the initial configuration the signed discrepancy is of the form $4k+2$ (for example, a single ligth on) then whatever the Player does the discrepancy will be at least two. Leo Moser conjectured that the Player can always achieve this minimal discrepancy. That is, given any initial configuration there is a set of row and column shifts that yield a configuration with discrepancy one, if n is odd, and with discrepancy either zero or two when n is even. In 1970, J. Komlós and M. Sulyok [3] succeeded in proving this result for all sufficiently large n. In this paper we give another proof of the Moser conjecture. Our proof, which is valid for all n, differs considerably from the original proof. It is algorithmic in the sense that from it one could readily construct

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an algorithm which, given an initial configuration, would determine the appropriate row and column shifts in polynomial time. In the late 1960's a succession of improvements were made on an upper bound to the discrepancy. In 1968 the senior author (J. S.) made one such improvement and discovered, the following week, that Komlós and Sulyok had resolved the full conjecture. It is satisfying, more than a decade later, to make a further contribution to this problem.

It will help to place the Moser Conjecture in a more numerical format. A configuration will be denoted by a matrix $A=(a_{ij})$ with all $a_{ij}=\pm 1$. The value $a_{ij}=+1$ corresponds to the $(i-j)$-th light being on and $a_{ij}=-1$ to the $(i-j)$-th light being off. The column shifts are denoted by variable $x_1, \dots, x_n=\pm 1$ where $x_j=+1$ denotes that the j-th column is not shifted and $x_j=-1$ denotes that the j-th column shifted. The row shifts are denoted similarly by variables $y_1, \dots, y_n=\pm 1$. (Note, critically, that line shifts are commutative and of order two. Hence any procedure followed by the Player can be reduced to a set, as opposed to a sequence, of line shifts and hence may be represented by the variables $x_1, \dots, x_n, y_1, \dots, y_n$.) Now we may restate Moser's Conjecture: Given any $n\times n$ matrix $A=(a_{ij})$ with all $a_{ij}=\pm 1$ there exist $x_1, \dots, x_n, y_1, \dots, y_n=\pm 1$ so that, setting

$$D = \left| \sum_{i=1}^{n} \sum_{j=1}^{n} y_i x_j a_{ij} \right|,$$

$D=1$, when n is odd, and $D\leqq 2$, when n is even.

We will let r_i, $1\leqq i\leqq n$, denote the i-th row sum of a configuration. Note r_i may be positive, negative, or zero. Our approach will be to find column shifts so that the r_i have an appropriate form and apply row shifts at the last step.

Lemma. *Given any initial configuration (a_{ij}) there exist column shifts x_j so that the new row sums r_i satisfy $|r_i|<2i$, $1\leqq i\leqq n$.*

We fix a_{ij}, write

$$r_i = \sum_{j=1}^{n} a_{ij} x_j, \quad 1 \leqq i \leqq n,$$

and think of the r_i as linear forms on variables $x_1, \dots, x_n$. We shall consider the x_j as real variables which will assume values in the closed interval $[-1, +1]$ during the argument though they must equal either $+1$ or -1 at the end of the argument. Initially all x_j are set equal to zero.

Call a variable x_j fixed if $x_j=\pm 1$ and floating if $-1<x_j<+1$. Once a variable is fixed it does not change. Suppose that at an intermediate stage there are exactly s floating variables (say $x_1, \dots, x_s$ for convenience) and that the first s row sums $r_1, \dots, r_s$ all equal zero. (Here "first s rows" is not simply for convenience. We assume a fixed ordering of the rows.) Consider the system of equations $r_i=0$, $1\leqq i\leqq s-1$ (critically, dropping the s-th row from consideration), $s-1$ linear equations in a variables $x_1, \dots, x_s$. (The remaining $x_j=\pm 1$ are treated as constants.) By linear algebra we find a line of solutions.

$$(x_1', \dots, x_s') = (x_1, \dots, x_s)+\lambda(c_1, \dots, c_s)$$

(with $x_1, \dots, x_s$ here representing the specific previous values) to this system. Geometrically, we move along this line until reaching the border of the s-cube. Algebraically, we set λ equal to that real of minimal absolute value such that some $x_t'=\pm 1$. Now replace the old x_j with the new x_j'. There are now at most $s-1$ floating variables (as some variable has hit the border) and the first $s-1$ row sums still equal zero.

Initially the condition "first s row sums equal zero where s is the number of of floating variables" is satisfied with $s=n$. Hence we may apply the above procedure at most n times to find n fixed variables $x_1, \ldots, x_n$. These x_j satisfy the Lemma. To see this, fix an arbitrary i and consider the value of the i-th row sum r_i as the procedure is carried out. At the first time in which there are at most i floating variables the row sum is still zero. After that these i variables x_j can float but each x_j changes less than two since it was in the open interval $(-1, +1)$ and ends up equal to either $+1$ or -1. A change of less than two in x_j yields a change of less than two in the row sum r_i since x_j is multiplied by a_{ij} and $|a_{ij}|\leqq 1$. As there are at most i floating variables the row sum changes less than $2i$ from its value zero and hence the Lemma is proven.

The implementation of this algorithm requires the calculation of $(c_1, \ldots, c_s)$, not all zero, satisfying the homogeneous system

$$\sum_{j=1}^{n} a_{ij}c_j = 0, \quad 1 \leqq i \leqq s-1.$$

The c_j may be found by basic linear algebra techniques in time $0(s^3)=0(n^3)$. As this procedure may be required n times the entire calculation may be done in time $0(n^4)$.

The techniques used in this Lemma have been studied in a more general setting. We refer the reader to [2] for comparable results.

We now jump ahead to describe a simple technique that will give the final row shifts. Let $s_1, \ldots, s_n$ be nonnegative integers and let K be a positive integer such that $s_1 \leqq K$ and so that for $2 \leqq i \leqq n$,

$$s_{i+1} \leqq s_1+\ldots+s_i+K.$$

Then there exist $y_1, \ldots, y_n=\pm 1$ so that

$$|y_1 s_1+\ldots+y_n s_n| \leqq K.$$

We find the y_i be reverse induction. Set $y_n=+1$. Having found $y_n, y_{n-1}, \ldots, y_{i+1}$ we choose $y_i=\pm 1$ so as to minimize the absolute value of the partial sum $y_n s_n+\ldots \ldots+y_{i+1}s_{i+1}+y_i s_i$. Our condition on the s_i assures that we never get "stuck" and that the final sum has the desired property. We shall call this method the Greedy Technique for the remainder of the paper. Our object will be to shift columns so that, setting $s_i=|r_i|$, the Greedy Technique may be applied to the s_i.

We now assume n is even and set $K=2$. Given an arbitrary configuration we apply the Lemma so that $|r_i|<2i$. For simplicity of notation let us then apply row shifts so that all row sums are nonnegative. Since all r_i are even integers and since the equality above is strict we have $r_1=0$, $r_2\in\{0, 2\}$, $r_3\in\{0, 2, 4\}$ and, in general, $r_i\leqq 2i-2$. We may not immediately apply the Greedy Technique because we may have too many $r_i=0$. For example, if the row sum sequence begins 0, 0, 0, 0, 0, 10, ... the Greedy Technique will not apply.

Reorder the rows in increasing order of row sums. We then still have $r_i \leqq 2i-2$. Suppose the first u rows have sum zero and the next v rows have sum two. If $u=1$ we may simply apply the Greedy Technique so we shall assume $u>1$. Let r_i' be the new absolute value of the i-th row sum after a single column is shifted. For the first u rows $r_i'=2$ regardless of which column is shifted. For the next v rows $r_i'=0$ for $(n/2)+1$ of the possible column shifts, these being the cases when an $a_{ij}=+1$ switched to -1, and $r_i'=4$ for the remaining $(n/2)-1$ column shifts. Thus the average value of r_i', taken over all n possible column shifts, is $4((n/2)-1)/n=2-4/n$. Now we use that the average of a sum is the sum of the averages and conclude that the average value of $r'_{u+1}+\ldots+r'_{u+v}$, the new row sums that had old row sums two, is $v(2-4/n)=2v-4v/n$. If half of the odd row sums equal two then the Greedy Technique trivially works and hence we may assume $v<n/2$. Thus $2v-4v/n>2v-2$. Since this is the average there must be one specific column change so that $r'_{u+1}+\ldots+$ $+r'_{u+v}>2v-2$ and since this sum is an even integer we have

(i) $\quad r'_{u+1}+\ldots+r'_{u+v} \geqq 2v.$

We also have

(ii) $\quad r_1' = \ldots = r_u' = 2.$

(iii) $\quad r_i' \leqq 2i, \quad i > u+v.$

These latter properties hold for any column shift since each row sum is changed by exactly two.

We observe that $r_1'+\ldots+r'_{u+v} \geqq 2(u+v)$ and $r_1' \geqq 2$ for $i>u+v$ since $r_i \geqq 4$. Hence

$$r_1'+\ldots+r_i'+2 \geqq 2i+2 \geqq r_{i+1}$$

for all $i \geqq u+v$. Trivially

$$r_1'+\ldots+r_i'+2 \geqq 4 \geqq r'_{i+1}$$

when $1 \leqq i < u+v$. Thus we may apply the Greedy Technique to the row sums $r_1', \ldots, r_n'$ completing the proof of the Moser Conjecture in the case n even.

To implement this portion of the algorithm we may simply check all n possible column shifts until we find one satisfying (i) above. The time contribution is negligible compared to the $0(n^4)$ of the previous step.

Now we turn to the case n odd. We assume $n \geqq 7$. If we follow directly the arguments of the n even case we may easily show that an arbitrary configuration can be shifted so as to give discrepancy $D \leqq 5$. To achieve $D=1$ requires a more detailed argument.

We first reexamine our Lemma and show that column shifts exist so that the new row sums r_i satisfy $|r_1| \leqq 1$ and $|r_i| < 2i-1$ for $2 \leqq i \leqq n$. The method for finding the x's remains the same. We use the fact that λ was that real of minimal absolute value so that some $x_t' = \pm 1$. (Geometrically, the solution line intersects the s-cube at two points and we are careful to select that point closest to our initial point.) Suppose $x_t' = \pm 1$, the opposite case being similar. Then the old value x_t must have been nonnegative for had x_t been negative we could have gone in the opposite direction — i.e., found a smaller λ of opposite sign. When bounding r_i there were at most i floating variables when r_i was last equal to zero. All of them float less than two and one of them (that particular x_t fixed at the next stage) floats at most one. Thus the i-th row sum r_i changes from zero by at most $2i-1$ and, when $i>1$, by strictly less than $2i-1$ since $i-1$ variables x_j float strictly less than two.

Applying this modification, and using the fact that r_i must be an odd integer, we may find a column shift so that $r_1=1$, $r_2=1$, $r_3\leqq 3$, and, for all $i>1$, $r\leqq 2i-3$. At this point application of the "average column shift" argument used in the n even case almost completes the proof of Moser's Conjecture. There is one stumbling block — in making a single column shift we may change all of the row sums that were one into three and have no new row sums equal to one. This would be the case, for example, if all of the rows were identical. Dealing with this case has proven cumbersome.

We are given an arbitrary configuration of odd order n. Suppose first that every two columns are either equal or negatives of each other. This is equivalent to saying that we may transform the configuration into the all $+1$ configuration. Shifting precisely $(n-1)/2$ columns and $(n-1)/2$ rows of the all $+1$ configuration gives a configuration with discrepancy $D=+1$. We may thus assume that there exist two columns, say the first two, whose corresponding coefficients are equal at least once and unequal at least once. Shifting one column if necessary (reversing equal and unequal) we may assume that the coefficients are unequal in at least half of the rows. Placing a row where they are equal first and following it by the rows in which they are unequal we may assume that

$$a_{11} = a_{12}$$

$$a_{i1}+a_{i2} = 0, \quad 2 \leqq i \leqq (n+3)/2.$$

Let B denote the submatrix consisting of the first $(n+3)/2$ rows and all but the first two columns and suppose, after appropriate line shifts, B has row sums $r_1, \ldots, r_{(n+3)/2}$. The first two columns of the full matrix will remain equal in the first row and unequal through row $(n+3)/2$. By either leaving these two columns both unshifted or by shifting both of them, and then shifting the first row if necessary, the full matrix is given row sums $s_1, \ldots, s_n$ where $s_1=|r_1-2|$ or $s_1=|r_1+2|$ and $s_i=r_i$ for $2\leqq i\leqq (n+3)/2$. For the remaining rows we need only $s_i\leqq n$. In particular, when $r_1=1$ we may either set $s_1=1$ or set $s_1=3$. It will suffice to find s_i that satisfy the Greedy Technique with $K=2$ (since the final discrepancy is odd). That is, we shall show $s_1, \ldots, s_{(n+3)/2}$ satisfy the Greedy Technique with $K=2$ and $s_1+\ldots+s_{(n+3)/2}+2\geqq n$.

We apply our Lemma to submatrix B so that its row sums are bounded by $1, 1, 3, 5, \ldots$ respectively Reorder the rows (but keep the first row, which is special, in its place) so that the row sums of B are in increasing order and suppose that the row sum sequence begins with u ones and v threes. If $u\leqq 3$ we may set $s_1=1$ and apply the Greedy Technique (the extreme row sum sequence would begin $1, 1, 1, 5, \ldots$) so we assume $u\geqq 4$.

We apply the "average column shift" argument to the first $u+v$ rows of B. A row sum $r_i=1$ has average value $2-1/(n-2)$ after shifting one of the $n-2$ columns of B. A row sum $r_i=3$ has average value $3-6/(n-2)$ after a column shift. Since $n\geqq 7$, $3-6/(n-2)\geqq 2-1/(n-2)$ and the $u+v$ rows have average sum at least $2(u+v)-(u+v)/(n-2)$ after a column shift. Now $u+v<n-2$ (unless $u+v=n-2=$ $=(n+3)/2=5$ in which case we set $s_1=1$ and apply the Greedy Technique) so this

average is more than $2(u+v)-1$. Thus there is a particular column shift after which the new row sums r_i' of B satisfy

(i) $r_1'+\ldots+r_{u+v}' \geqq 2(u+v)$

(ii) $r_i' \leqq 2i-1, \quad i>1$

(iii) $r_i' \geqq 3, \quad i>u+v$

and so that the first four row sums (which were one) are either one or three.

If $r_1'=1$ we set $s_1=1$ and apply the Greedy Technique. When $r_1'=3$ we have a choice of setting $s_1=1$ or $s_1=5$. If there is at least one $r_i'=1$ we set $s_1=5$ and apply the Greedy Technique. (For example, 3, 1, 1, 1, ... becomes 5, 1, 1, 1, ... and 3, 1, 3, 3, ... becomes 5, 1, 3, 3, ...) Otherwise the sequence r_i' begins 3, 3, 3, 3. In this case we set $s_1=1$ and the Greedy Technique again applies. This completes the proof of the Moser Conjecture for all n except $n=3$ and $n=5$. These values may be easily checked by hand.

References

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